Chapter 11. Integrable Highest - weight Modules : the weight system and the Unitarizability.

§11.1.

· Fix $\Lambda \in P_+$, compare to prop 3.6 P34.

prop 11.1. Let $\lambda \in P(\Lambda)$, $\alpha \in \Delta^{re}$ and $m_t = \text{mult}_{L(\Lambda)}(\lambda + t\alpha)$, Then.

a) The the set of $t \in \mathbb{Z}$, such that $\lambda + t\alpha \in P(\Lambda)$ is the interval $\{t \in \mathbb{Z} \mid -p \le t \le q\}$, where p and q are nonnegative integers and $p - q = \langle \lambda, \alpha^\vee \rangle$

b) For $e_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$, the map $e_\alpha : L(\Lambda)_{\lambda + t\alpha} \longrightarrow L(\Lambda)_{\lambda + (t+1)\alpha}$ is an injection if $-p \le t < \frac{1}{2}(q - p)$; in particular, the function $t \longmapsto m_t$ increases on this interval.

c) The function $t \longmapsto m_t$ is symmetric w.r.t. $t = \frac{1}{2}(q - p)$

d) If both λ and $\lambda + \alpha$ are weights, then $\mathfrak{g}_\alpha(L(\Lambda)_\lambda) \neq 0$.

proof: By Lem 10.1: The $\mathfrak{g}(A)$-module $L(\Lambda)$ is integrable if $\Lambda \in P_+$ and results of prop 3.6 b). for simple root α, the prop. holds

W 群定义. Nichols diagonal 链 ~~[illegible]~~?

· Applying prop 3.7 a) ; let V be an integrable module over a Kac-Moody algebra $\mathfrak{g}(A)$. Then $\text{mult}_V \lambda = \text{mult}_V w(\lambda)$ for every $\lambda \in \mathfrak{h}^*$, and $w \in W$. ☆
b) The root system Δ of $\mathfrak{g}(A)$ is W-invariant, and $\text{mult}\,\alpha = \text{mult}\,w(\alpha)$. for every $\alpha \in \Delta$, $w \in W$.

⌈ since $\forall \alpha \in \Delta^{re}$, $\exists\, w \in W$, s.t. $w(\alpha) = \alpha_i$ (simple root). i.e. $\exists\, w' \in W$, s.t. $w'(\alpha_i) = \alpha$. since $P(\Lambda)$ is W-invariant, i.e. $w(\lambda) \in P(\Lambda)$. then $\lambda' \in P(\Lambda)$. s.t. $w'(\lambda') = \lambda$ $\Rightarrow$ $\text{mult}_{L(\Lambda)}(\lambda + t\alpha) = \text{mult}_{L(\Lambda)}(w'(\lambda' + t\alpha_i)) = \text{mult}_{L(\Lambda)}(\lambda' + t\alpha_i)$

verify $p - q = \langle \lambda', \alpha_i^\vee \rangle = \langle \lambda, \alpha^\vee \rangle$.

$$\langle \lambda', \alpha_i^\vee \rangle = (\lambda' \mid \nu(\alpha_i^\vee)) = \left(\lambda' \,\Big|\, \frac{2\alpha_i}{(\alpha_i|\alpha_i)}\right), \qquad \langle \lambda, \alpha^\vee \rangle = (\lambda \mid \nu(\alpha^\vee)) = \left(\lambda \,\Big|\, \frac{2\alpha}{(\alpha|\alpha)}\right)$$
⌋

§11.2.

· Fix $\Lambda \in P_+$, Recall that $P(\Lambda)$ is W-invariant

· An element $\lambda \in P$ is called nondegenerate with respect to Λ if either $\lambda = \Lambda$ or else $\lambda < \Lambda$ and for every connected component S of $\text{supp}(\Lambda - \lambda)$ one has :

$$S \cap \{i \mid \langle \Lambda, \alpha_i^\vee \rangle \underset{>0}{\neq} 0\} \neq \emptyset \qquad (11.2.1)$$

Pbr. Recall : For $\alpha = \sum k_i \alpha_i \in Q$, we define the support of α (written $\text{supp}\,\alpha$) to be the subdiagram of $S(A)$ which consists of the vertices i such that $k_i \neq 0$ and of all the edges joining these vertices.

Lem 11.2 Every weight λ of the $g(A)$-module $L(\Lambda)$ is nondegenerate w.r.t. Λ.

proof: Suppose that $\lambda \in P(\Lambda)\setminus\{\Lambda\}$, let S be a connected component of $\mathrm{supp}(\Lambda-\lambda)$, Denote by $n_-(S)$ the subalgebra of n_- generated by f_i, s.t. $i \in S$.

Then: (11.2.2) $L(\Lambda)_\lambda \subset U(n_-) n_-(S) \underline{L(\Lambda)_\Lambda}$

If (11.2.1) were false, then $n_-(S) L(\Lambda)_\Lambda = 0$. for some S

[$f_i(L(\Lambda)_\Lambda) \subset L(\Lambda)_{\Lambda-\alpha_i}$, $r_i(\Lambda) = \Lambda - \underline{\langle \Lambda, \alpha_i^\vee\rangle}\alpha_i = \Lambda$]

hence $L(\Lambda)_\lambda = 0$. Contradiction #

irreducible ? integrable

prop 11.2. Let $\Lambda \in P_+$, Then.

a). $P(\Lambda) = W \cdot \{\lambda \in P_+ \mid \lambda \text{ is nondegenerate with respect to } \Lambda\}$.

b) If $\underline{\{i \mid \langle \Lambda, \alpha_i^\vee\rangle = 0\} \subset S(A)}$ is a disjoint union of diagrams of finite type, then

$$P(\Lambda) = W \cdot \{\lambda \in P_+ \mid \lambda \leq \Lambda\}$$

proof: The inclusion "$\subset$" in a) and b) follows from Cor. 10.1 and Lem 11.2.

[Recall: Cor 10.1. If $\Lambda \in P_+$, then any $\lambda \in P(\Lambda)$ is W-equivalent to a unique $\mu \in P_+ \cap P(\Lambda)$]

The other inclusion in b) follows from that of a).
Indeed, if $\lambda \in P_+$ and $\lambda = \Lambda - \beta$, where $\beta \in Q_+$, then $\langle \beta, \alpha_i^\vee\rangle \leq 0$ for i such that $\langle \Lambda, \alpha_i^\vee\rangle = 0$.

[$\langle \lambda, \alpha_i^\vee\rangle = \langle \Lambda - \beta, \alpha_i^\vee\rangle = -\langle \beta, \alpha_i^\vee\rangle \geq 0$.
$\forall S \subset \{i \mid \langle \Lambda, \alpha_i^\vee\rangle = 0\}$ is a finite type. $\forall i \in S$, $\langle \beta, \alpha_i^\vee\rangle \leq 0$.
$\exists u$, s.t. $Au \leq 0$.]

$\cdot\ \alpha' = \sum_{i\in S} k_i \alpha_i$, $\langle \alpha', \alpha_i^\vee\rangle \leq 0 \Rightarrow (\alpha'|\alpha') \leq 0$

which implies that λ is nondegenerate w.r.t. Λ.

. It remains to show that if $\mu = \Lambda - \alpha \in P_+$, where $\alpha = \sum_i k_i \alpha_i$, $k_i \geq 0$, $\sum_i k_i > 0$, and $S \cap \{i \mid \langle \Lambda, \alpha_i^\vee\rangle \neq 0\} \neq \phi$ for every connected component S of $\mathrm{supp}\,\alpha$, then $\underline{\mu \in P(\Lambda)}$. (i.e. "$\supset$" of a)).

Let: $\Omega_\alpha = \{\gamma \in Q_+ \mid \gamma \leq \alpha \text{ and } \Lambda - \gamma \in P(\Lambda)\}$. Then $\underline{\text{set } \Omega_\alpha \text{ is finite}}$. (Pf: Lem. 5.3).
and the union of supports of its elements has a nonempty intersection with each connected component of $\mathrm{supp}\,\alpha$.
(i.e. $(\bigcup_{\gamma\in\Omega_\alpha} \mathrm{supp}(\gamma)) \cap S \neq 0$ where S is any connected comp. of $\mathrm{supp}\,\alpha$.).

Let $\beta = \sum_i m_i \alpha_i$ be an element of maximal height in Ω_α.
It follows from from Prop 11.1 a) that:
(11.2.3) $\mathrm{supp}\,\beta = \mathrm{supp}\,\alpha$.

[If $\underline{\text{some}}$ $i \in \mathrm{supp}\,\alpha \setminus \mathrm{supp}\,\beta$, but $\langle \beta, \alpha_i^\vee\rangle < 0$. $\Rightarrow \beta + \alpha_i \in \Omega_\alpha$. contradicts to maximal] suppose that $\beta \neq \alpha$. Then

$$(11.2.4) \quad \Lambda - \beta - \alpha_i \notin P(\Lambda) \quad \text{if} \quad k_i > m_i.$$

Set $S = \{ j \in S(A) \mid k_j = m_j \}$. Let R be a connected component of $(\mathrm{supp}\,\alpha) \setminus S$, we deduce from (11.2.4) and prop 11.1 a)

$$(11.2.5) \quad \langle \beta, \alpha_i^\vee \rangle \geq \langle \Lambda, \alpha_i^\vee \rangle \quad \text{and} \quad \langle \alpha, \alpha_i^\vee \rangle \leq \langle \Lambda, \alpha_i^\vee \rangle \quad \text{if } i \in R$$

[Since $\Lambda - \alpha + \alpha_i$ is weight].

[Cor 3.6. a) If $\lambda \in P(\Lambda)$, and $\lambda + \alpha_i \notin P(\Lambda)$ (resp $(\lambda - \alpha_i) \notin P(\Lambda)$) $\Rightarrow$ $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (resp ≤ 0).]

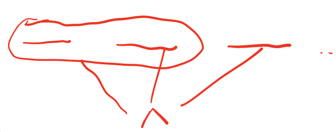
Set $\beta' = \sum_{i \in R} m_i \alpha_i$, $\alpha' = \sum_{i \in R} (k_i - m_i) \alpha_i$. Then (11.2.3) and (11.2.5) imply:

$$(11.2.6) \quad \langle \beta', \alpha_i^\vee \rangle \geq 0 \quad \text{if } i \in R. \quad (\text{by cor. } 3.6).$$

$$(11.2.7) \quad \langle \alpha', \alpha_i^\vee \rangle \leq 0 \quad \text{if } i \in R \quad (\text{by } (11.2.5).)$$

It follows from (11.2.7) that R and hence $S(A)$ are not of finite type [$\Rightarrow (\alpha' | \alpha') \leq 0$]. In particular, for every $\lambda \in P(\Lambda)$, there exists α_i such that $\lambda - \alpha_i \in P(\Lambda)$. (otherwise $\dim(L(\Lambda)) < \infty$ and $\dim \mathfrak{g}(A) < \infty$). Hence $S \neq \emptyset$ and by the properties of μ, we can choose R so that it is not a connected component of $\mathrm{supp}\,\alpha$. [$\Lambda - \beta \in P(\Lambda)$, $\Lambda - \beta - \alpha_i \in P(\Lambda)$.] But then, in addition to (11.2.6) we have: $\langle \beta', \alpha_j^\vee \rangle > 0$ for some $j \in R$.

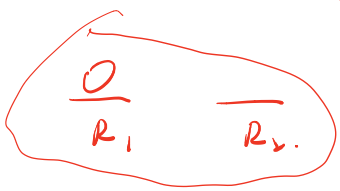
supp α supp $\alpha \setminus S$



$\beta' - \alpha_j \in$

β'

$\langle \Lambda, \alpha_i^\vee \rangle > 0$.



$\langle \beta', \alpha_i^\vee \rangle \geq 0$ $i \in R_1$

$\langle \beta', \alpha_j^\vee \rangle \geq 0$ $j \in R_2$.

if $\langle \beta', \alpha_j^\vee \rangle = 0$.

$S \neq \emptyset$

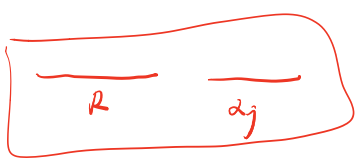
$k_i > m_i$

α_i.

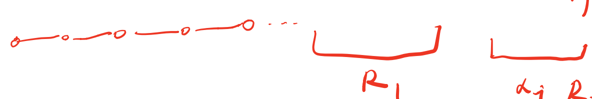
$\beta' = \sum_{i \in R} m_i \alpha_i$.

α_j

$k_j = m_j$



supp α.



$\beta > \beta'$